# COL727: Rapid Mixing in Markov Chains
I semester, 2024-25
Minor I
Total Marks 100
**Due: On Gradescope at 11:55PM, 11th August 2024**

**Instructions:** 1. Please write legibly and make sure your scan is readable. 2. Remember to map problem numbers to pages of your pdf on Gradescope. 3. Be succinct in your answers but explain your calculations in words.

**Problem 1** ($30 + 15 + 15 = 60$ **marks**) *We are given a Markov chain $(X_t)_{t \geq 0}$ on a finite set $\mathcal{X}$ with stationary distribution $\pi$. First prove the result of Problem 1.1, then use the result to prove the results given in Problems 1.2 and 1.3. For the notation $\tau_x, \tau_x^+$ please see Section 1.5.2 of LPW.*

**Problem 1.1** (30 **marks**) *Suppose $\mu$ is a probability distribution on $\mathcal{X}$. Let $T > 0$ be a stopping time such that $\mathbb{P}_\mu \{X_T = \cdot\} = \mu(\cdot)$, i.e., the distribution at time $T$ is the same as the distribution at time 0, and $\mathbb{E}_\mu \{T\} < \infty$. Also $T < \infty$ with probability 1. Suppose $x$ is an arbitrary state from $\mathcal{X}$, show that*

$$\mathbb{E}_\mu \{\textit{Number of visits to } x \textit{ (strictly) before time } T\} = \pi(x)\mathbb{E}_\mu \{T\}.$$

*(Hint: This is a generalization of Prop 1.14 of LPW.)*

**Problem 1.2** (15 **marks**) *For states $x \neq z \in \mathcal{X}$ and $y \in \mathcal{X}$,*

$$\mathbb{E}_x \{\textit{Number of visits to } y \textit{ before } \tau_z\} = \pi(y) \left(\mathbb{E}_x \{\tau_z\} + \mathbb{E}_z \{\tau_y\} - \mathbb{E}_x \{\tau_y\}\right)$$

**Problem 1.3** (15 **marks**) *For any states $x \neq y \in \mathcal{X}$,*

$$\mathbb{P}_x \left\{\tau_y < \tau_x^+\right\} = \frac{1}{\pi(x) \left(\mathbb{E}_x \{\tau_y\} + \mathbb{E}_y \{\tau_x\}\right)}.$$

**Problem 2** (40 **marks**) *Suppose we have a Markov Chain with transition matrix $P$ on finite state space $\mathcal{X}$ and we have two distributions $\mu$ and $\nu$ on $\mathcal{X}$, show that*

$$\|\mu P - \nu P\|_{TV} \leq \|\mu - \nu\|_{TV}.$$

**Note.** *This can be proved easily by manipulating sums but you must prove it using coupling* without *any calculation whatsoever in order to get non-zero marks.*