# COL727: Rapid Mixing in Markov Chains
## I semester, 2024-25
## Minor II
Total Marks 100

**Due: On Gradescope at 11:55PM, 8th September 2024**

**Instructions:** 1. Please write legibly and make sure your scan is readable. 2. Remember to map problem numbers to pages of your pdf on Gradescope. 3. Be succinct in your answers but explain your calculations in words. 4. *In this exam* **page counts** *will be enforced. Use the number of pages allowed per question. Write/latex on standard A4 sheets in a reasonable sized font ($\geq$ 11pt for latex users). Only the given number of pages will be read in your scan*

**Problem 1 (60 marks. Maximum page count: 2)** *Suppose we have a Markov chain on $\mathcal{X} = [n] \times [n]$ for some $n > 1$ that makes transitions to the $\ell_1$ neighbour of each state uniformly, i.e., from $(x, y)$ the state goes with $1/4$ probability to $(x \pm 1, y \pm 1)$, except if $x$ or $y$ or both are either 1 or $n$, in which case the chain goes to each neighbour with probability $1/4$ and loops with the remaining probability.*

*Give an upper bound using coupling for this chain. Note, you* must *use coupling. Give a lower bound as well using any method you feel is appropriate.*

**Problem 2 (40 marks. Maximum page count: 3)** *In this problem we will try to do a direct analysis of a version of the Metropolis chain on $q$-colourings,* without *using the metric contraction method that was discussed in class. As before we have a graph $G = (V, E)$ with $|V| = n$ and maximal degree $\Delta$. This version of the Metropolis chain works in rounds: each vertex is chosen for update exactly once in each round. The order within a round is decided by choosing a random permutation of $V$. The permutation chosen for each round is independent of all other rounds. This gives rise to an order: $v_1, \ldots, v_n, v_{n+1}, \ldots, v_{2n}, \ldots$*

*The Metropolis rule is now modified slightly. Given a colouring $X_t$ at step $t$, the Metropolis chain with $q$ colours selects the vertex $v_t$ according to the order defined above and a colour $c$ u.a.r. from $[q]$. If $c$ doesn't exist in the neighbours of $v_t$ in $X_t$, then $v_t$ gets recoloured, otherwise the chain does nothing.*

*Use the result of Corollary 5.5 to get a bound on $t_{mix}$ for this chain, i.e., beginning with two* proper *colourings $\boldsymbol{x}$ and $\boldsymbol{y}$, couple these chains as appropriate and then upper bound the probability that the two chains have not coalesced at time $t$ to get a bound on $d(t)$. You are free to choose $q$ to be larger than $3\Delta$ if you like but you are not allowed to choose a value of $q$ which grows as $n$ grows. (Hint: Something like $q > 2\Delta^2$ might work)*