COL727: Rapid Mixing in Markov Chains
I semester, 2024-25
Minor IV

Total Marks 100

Due: On Gradescope at 11:55PM, 14th November 2024

Instructions: 1. Please write legibly and make sure your scan is readable. 2. Remember to map problem numbers to pages of your pdf on Gradescope. 3. Be succinct in your answers but explain your calculations in words. 4. *In this exam* **page counts** *will be enforced. Use the number of pages allowed per question. Write/latex on standard A4 sheets in a reasonable sized font (≥ 11pt for latex users). Only the given number of pages will be read in your scan*

Problem 1 ($20 + 30 = 50$ marks. Total page count: 5) *Consider the following scenario: We have a connected graph $G = (V, E)$ where $|V| = n$ whose vertices have names $v_1, \ldots, v_n$. We call v_n the* sink. *At $t = 0$ all the vertices of $V \setminus \{v_n\}$ have a pebble. Each of these pebbles starts executing a simple random walk on G. The moment a pebble reaches the sink it is removed from the network, i.e., we have $n-1$ processes $\{X_t^{(i)}\}_{t\geq 0}$, where $X_0^{(i)} = v_i$ for $1 \leq i \leq n-1$. We are interested in the "collection time" for these pebbles, i.e.,*

$$\tau_c = \max_{i=1}^{n-1} \min\{t : X_t^{(i)} = v_n\}.$$

In words this is the time that the last of the pebbles is removed. There are two problems associated with this setting below.

Problem 1.1 (20 marks. Max page count: 2) *First, let us consider the scenario where all the $n-1$ processes are independent Markov chains. Find a bound T such that for $\epsilon \in (0, 1)$, $P\{\tau_c > T\} \leq \varepsilon$.*

Problem 1.2 (30 marks. Max page count: 3) *Now we consider a "queueing" scenario, i.e., we view each vertex apart from v_n as a queue that can receive multiple pebbles in each time step but can send only one pebble out at a time. We apply the rule that the pebble whose originating vertex has the smallest id gets priority. In other words, if for some $t > 0$ for some vertex v, if $X_t^{(i_1)} = X_t^{(i_2)} = \cdots = X_t^{(i_k)} = v$ with $i_1 < i_2 < \cdots i_k$ then $X_{t+1}^{(i_2)} = \cdots = X_{t+1}^{(i_k)} = v$ while $X_{t+1}^{(i_1)}$ may be some neighbour of v according to the rules of the simple random walk on G. Find a bound T such that for $\epsilon \in (0, 1)$, $P\{\tau_c > T\} \leq \varepsilon$. (Warning: It's not straightforward to get a good bound for this problem. Hint: Trace time backwards from the time when the last pebble reaches the sink. The first time in this backward path that you find it was delayed, start following backward in time the pebble that had led to the delay. Keep doing this till you reach a pebble that is at its starting point. Now analyse this trajectory.)*

Problem 2 (20 marks. Max page count: 2) *Exercise 14.8 of LPW.*

Problem 3 (30 marks. Max page count: 2) *Exercise 13.12 of LPW. Note that you may need to read some parts of Chapter 13 that were not covered in class.*