COL727. L20. 18.10.2024 (1)

2 A Laplacian associated with ~~the~~ the random walk on a network

Let us consider a network on $\mathcal{X}$ with positive weights $c : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$. (conductances). For each $x \in \mathcal{X}$, $d(x) = \sum_{y \in \mathcal{X}} c(x,y)$.

A is the adjacency matrix, ie, $A(x,y) = c(x,y)$
[we assume $c(x,x) = 0$ for all x]

D is a diagonal matrix with $D(x,x) = d(x)$.

The Laplacian associated with the network is

$$L = D - A.$$

Claim: L is positive semi definite. ~~all eigen~~ The smallest eigen value of L, $\lambda_0 = 0$ and is associated with the vector $1 = (1, \dots 1)$.

Proof: $$a^T L a = \sum_{x \in \mathcal{X}} d(x) a(x)^2 - \sum_{x \in \mathcal{X}} \sum_{\substack{y \in \mathcal{X} \\ y \neq x}} a(x) a(y) c(x,y)$$

Noting that $d(x) = \sum_{y \in \mathcal{X}} c(x,y)$ and $c(x,y) = c(y,x)$,

$$a^T L a = \sum_{\substack{x,y \in \mathcal{X} \\ x \neq y}} c(x,y) \left[a(x)^2 + a(y)^2 - 2 a(x) a(y) \right]$$

$$= \sum_{\substack{x,y \in \mathcal{X} \\ x \neq y}} c(x,y) \left[a(x) - a(y) \right]^2 \geq 0$$

Since L is symmetric, the smallest e.val of L is $\min_{a \in \mathbb{R}^{\mathcal{X}}} \frac{a^T L a}{a^T a}$

and since $a^T L a \geq 0$, L is p.s.d. Also, when $a = 1$ then $a^T L a = 0$.

~~Define $L_P = D^{-1} [D - A] = I - P$ as the Laplacian of P.~~
~~By Prop 12.1 of LPW the e.vals of this are between 0 and 2 [0 and 1 if P is a lazy chain]. Also 0 is an e.val. of L_P associated with 1.~~
~~Note that if λ is an eigenvalue of P then $1 - \lambda$ is an eigenvalue of L_P. Therefore if we want to find the spectral gap $1 - \lambda_2$ of P we can look for the 2nd smallest e.val. of L_P~~

We "normalize" L as follows

$$N := D^{-1/2} L D^{-1/2} \quad = I - D^{-1/2} A D^{-1/2}$$

Note that $x^T N x = x^T D^{-1/2} L D^{-1/2} x$

and so N is also p.s.d. and 0 is an e.val associated with the vector $d^{1/2}$ [$\because d^{1/2} D^{-1/2} = 1$].

Let the eigenvalues of N be $0 = \nu_1 \le \nu_2 \cdots \nu_n$.

Now we come to the lazy random walk. Note that $P = AD^{-1}$, so the lazy walk matrix

$$W = \frac{1}{2}\left[I + AD^{-1}\right]$$

Note that
$$W = I - \frac{1}{2}\left[I - AD^{-1}\right]$$

$$= I - \frac{1}{2} D^{1/2}\left[I - D^{-1/2} A D^{-1/2}\right] D^{-1/2}$$

$$= I - \frac{1}{2} D^{1/2} N D^{-1/2}$$

$\therefore$ If ψ is an e.vec of N with e.val ν

$$W\left[D^{1/2}\psi\right] = \left[I - \frac{1}{2} D^{1/2} N D^{-1/2}\right] D^{1/2}\psi$$

$$= D^{1/2}\psi - \frac{1}{2} D^{1/2} N \psi$$

$$= \left(1 - \frac{\nu}{2}\right) D^{1/2} \nu.$$

$\therefore$ The (absolute) spectral gap of $W = \frac{\nu_2}{2}$

We can use such an upper bound to lower bound the mixing time of the walk.

By the Courant-Fischer theorem ~~and the fact that~~ we have

$$\nu_2 = \min_{x \perp d^{1/2}} \frac{x^T D^{-1/2} L D^{-1/2} x}{x^T x}$$

Using the transformation $x = D^{1/2} y$ this is the same as

$$\nu_2 = \min_{y \perp d} \frac{y^T L y}{y^T D y} \qquad \text{———} (**)$$

$\therefore$ To find ν_2 we can find a vector y s t $\sum_{x \in X} y(x) = 0$ and

compute $$\frac{\sum_{w, x \in X} c(w,x) \left[y(w) - y(x) \right]^2}{\sum_{x \in X} d(x)\, y(x)^2}$$

to get an upper bound on ν_2.

Example: Consider a line with n vertices and unit conductance on each edge. Let $y(x) = +1$ for $1 \le x \le n/2$ and -1 for the rest.

$$\text{Then } \nu_2 \le \frac{4}{2(n-2)+2} = \Theta\left(\frac{1}{n}\right)$$

$\therefore$ mixing time $\Omega(n)$ [weak bound]

Now try $y =$ $-n/2$, $-n/2+1$, ..., 1, 0, 1, ..., $n/2$

$$\nu \le \frac{n}{4\sum_{i=1}^{n/2} i^2} = \Theta\left(\frac{1}{n^2}\right) \text{ which gives } t_{mix} = \Omega(n^2)$$

(tight)

□