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Laplacian associated with the random walk on a network

①

Let us consider a network on X with positive weights $c: X \times X \rightarrow \mathbb{R}_+$ (conductances). For each $x \in X$, $d(x) = \sum_{y \in X} c(x, y)$.

A is the adjacency matrix, i.e., $A(x, y) = c(x, y)$
[we assume $c(x, x) = 0$ for all x]

D is a diagonal matrix with $D(x, x) = d(x)$.

The Laplacian associated with the network is

$$L = D - A.$$

Claim: L is positive semi-definite. ~~is the~~ The smallest eigen value of L , $\lambda_0 = 0$ and is associated with the vector $\mathbf{1} = (1, \dots, 1)$.

Proof: $a^T L a = \sum_{x \in X} d(x) a(x)^2 - \sum_{x \in X} \sum_{\substack{y \in X \\ y \neq x}} a(x) a(y) c(x, y)$

Noting that $d(x) = \sum_{y \in X} c(x, y)$ and $c(x, y) = c(y, x)$,

$$a^T L a = \sum_{\substack{x, y \in X \\ x \neq y}} c(x, y) [a(x)^2 + a(y)^2 - 2a(x)a(y)]$$

$$= \sum_{\substack{x, y \in X \\ x \neq y}} c(x, y) [a(x) - a(y)]^2 \geq 0$$

\Rightarrow Since L is symmetric, the smallest e. val of L is $\min_{a \in \mathbb{R}^X} \frac{a^T L a}{a^T a}$

and since $a^T L a \geq 0$, L is p.s.d. Also, when $a = \mathbf{1}$ then $a^T L a = 0$.

Define ~~L_P~~ $L_P = D^{-1} [D - A] = I - P$ as the Laplacian of P .

By Prop 12.1 of LPW the e. vals of this are between 0 and 2 [0 and 1 if P is a lazy chain]. Also 0 is an e. val. of L_P associated with $\mathbf{1}$.

Note that if λ is an eigenvalue of P then $1 - \lambda$ is an eigenvalue of L_P . Therefore if we want to find the spectral gap $1 - \lambda_2$ of P we can look for the 2nd smallest e. val. of L_P .

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We "normalize" L as follows

$$N := D^{-1/2} L D^{-1/2} = I - D^{-1/2} A D^{-1/2}$$

$$\text{Note that } x^T N x = x^T D^{-1/2} L D^{-1/2} x$$

and so N is also p.s.d. and 0 is an e.val associated with the vector $d^{1/2}$ $\{\because d^{1/2} D^{-1/2} = 1\}$.

Let the eigenvalues of N be $0 = \nu_1 \leq \nu_2 \leq \dots \leq \nu_n$.

Now we come to the lazy random walk. Note that $P = AD^{-1}$, so the lazy walk matrix

$$W = \frac{1}{2} [I + AD^{-1}]$$

$$\text{Note that } W = I - \frac{1}{2} [I - AD^{-1}]$$

$$= I - \frac{1}{2} D^{1/2} [I - D^{-1/2} A D^{-1/2}] D^{-1/2}$$

$$= I - \frac{1}{2} D^{1/2} N D^{-1/2}$$

\therefore If ψ is an e.vec of N with e.val ν

$$W [D^{1/2} \psi] = [I - \frac{1}{2} D^{1/2} N D^{-1/2}] D^{1/2} \psi$$

$$= D^{1/2} \psi - \frac{1}{2} D^{1/2} N \psi$$

$$= (1 - \frac{\nu}{2}) D^{1/2} \psi.$$

\therefore The (absolute) spectral gap of $W = \frac{\nu_2}{2}$

We can use such an upper bound to lower bound the mixing time of the walk.

By the Courant-Fischer theorem ~~and the fact that~~
we have

$$\nu_2 = \min_{x \perp d^{1/2}} \frac{x^T D^{-1/2} L D^{-1/2} x}{x^T x}$$

Using the transformation ~~$y = D^{1/2} x$~~ $x = D^{1/2} y$ this is
the same as

$$\nu_2 = \min_{y \perp d} \frac{y^T L y}{y^T D y} \quad \text{--- (**)}$$

\therefore To find ν_2 we can find a vector y st $\sum_{x \in X} y(x) = 0$ and

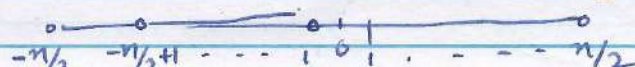
compute
$$\frac{\sum_{\omega, x \in X} c(\omega, x) [y(\omega) - y(x)]^2}{\sum_{x \in X} d(x) y(x)^2}$$

to get an upper bound on ν_2 .

Example: Consider a line with n vertices and unit conductance on each edge. Let $y(x) = +1$ for $1 \leq x \leq n/2$ and -1 for the rest.

$$\text{Then } \nu_2 \leq \frac{4}{2(n-2)+2} = \Theta\left(\frac{1}{n}\right)$$

\therefore mixing time $\Omega(n)$ [weak bound]

Now try $y =$ 

$$\nu \leq \frac{n}{4 \sum_{i=1}^{n/2} i^2} = \Theta\left(\frac{1}{n^2}\right) \text{ which gives } t_{\text{mix}} = \Omega(n^2) \text{ (tight)}$$

□