First Order Predicate Logic

Lecture 4

First Order Predicate Logic

Limitation of Propositional Logic

- The facts:
  - "peter is a man", "paul is a man", "john is a man" can be symbolized by P, Q and R respectively in propositional logic,
- Can't draw any conclusions about similarities between P, Q and R.
- Better to represent these facts as
  - MAN(peter), MAN(paul) and MAN(john).

Cont…

- Even more difficult to represent sentences like “All men are mortal” in propositional logic.
  - Such sentences really need quantification.
- In Predicate Logic, these limitations are removed to great extent.
- **Predicate Logic** is logical extension of propositional logic.
- First Order Predicate Logic is one where the quantification is over simple variables.

Predicate Calculus

- It has three more logical notions as compared to propositional calculus.
  - Terms
  - Predicates
  - Quantifiers (universal or existential quantifiers i.e. “for all' and “there exists”)
- **Term** is
  - a constant (single individual or concept i.e.,5,john etc.),a variable that stands for different individuals,
  - a function: a mapping that maps n terms to a term i.e., if f is n-place function symbol and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is a term.

Cont…

- **Predicate** : a relation that maps n terms to a truth value true (T) or false (F).
  – LOVE (john , mary)
  – LOVE(father(john), john)
  – LOVE is a predicate. father is a function.
- **Quantifiers**: Variables are used in conjunction with quantifiers.
  – There are two types of quantifiers viz.., “there exist” ($\exists$) and “for all” ($\forall$).
  – “every man is mortal” can be represented as ($\forall$x) (MAN(x) $\rightarrow$ MORTAL(x)).

Examples

- A statement "*x is greater than y*" is represented in predicate calculus as GREATER(x, y).
- It is defined as follows:

$$\text{GREATER}(x, y) = \begin{cases} T, & \text{if } x > y \\ F, & \text{otherwise} \end{cases}$$

- The predicate names GREATER takes two terms and map to *T* or *F* depending upon the values of their terms

Examples – Cont…

- A statement *"john loves everyone"* is represented as
  - $(\forall x)$ LOVE(john , x) which maps it to true when x gets instantiated to actual values.
- A statement *"Every father loves his child"* is represented as
  - $(\forall x)$ LOVE(father(x), x).
  - Here *father* is a function that maps *x* to his father.
- The predicate name LOVE takes two terms and map to *T* or *F* depending upon the values of their terms.

First Order Predicate Calculus

- The first order predicate calculus (FOPC) is a formal language.
  - Basic rules for formula in Predicate Calculus are same as those of Propositional Calculus.
  - A wide variety of statements are expressed in contrast to Propositional Calculus

Well-formed Formula

- **Well-formed formula** in FOPC is defined recursively as follows:
  - Atomic formula $P(t_1, \ldots, t_n)$ is a well-formed formula, where P is a predicate symbol and $t_1,\ldots,t_n$ are the terms. It is also called atom.
  - If α and β are well-formed formulae, then $\sim(\alpha)$, $(\alpha \vee \beta)$, $(\alpha \wedge \beta)$, $(\alpha \rightarrow \beta)$ and $(\alpha \leftrightarrow \beta)$ are well-formed formulae.
  - If α is a well-formed formula and x is a free variable in α, then $(\forall x)\alpha$ and $(\exists x)\alpha$ are well-formed formulae.
  - Well-formed formulae are generated by a finite number of applications of above rules.

Example

Example: Translate the text *"Every man is mortal. John is a man. Therefore, John is mortal"* into a FOPC formula.

Solution: Let MAN(x), MORTAL(x) represent that x is a man and x is mortal respectively.

- Every man is mortal : $(\forall x)\ (MAN(x) \rightarrow MORTAL(x))$
- John is a man : $MAN(john)$
- John is mortal : $MORTAL(john)$

The whole text can be represented by the following formula.

$$(\forall x)\ ((MAN(x) \rightarrow MORTAL(x))\ \Lambda\ MAN(john)) \rightarrow MORTAL(john)$$

First Order Predicate Logic

- First order predicate calculus becomes First Order Predicate Logic if inference rules are added to it.
- Using inference rules one can derive new formula using the existing ones.
- **Interpretations of Formulae in Predicate Logic**
  - In propositional logic, an interpretation is simply an assignment of truth values to the atoms.
  - In Predicate Logic, there are variables, so we have to do more than that.

Interpretation

- An *interpretation* of a formula α in FOL consists of
  - a non empty domain D and
  - an assignment of values to each constant, function symbol and predicate symbol occurring in α.
- It is denoted by I and is defined as follows:
  - Assign a value to each constant from the domain D.
  - Each n-place function f (mapping from D^n to D) is assigned a value from D such as $f(x_1, \ldots, x_n) = x$, where $(x_1, \ldots, x_n) \in D^n$ and $x \in D$.
  - Assign a value from a set {T, F} to each n-place predicate P (mapping from D^n to {T, F}). Here T represents *true* value and F represents *false* value.

Interpretation – Cont…

- The quantifiers $(\forall x)$ and $(\exists x)$ are interpreted as follows:
  - $(\forall x)$ will be interpreted as "for all elements x such that $x \in D$"
  - $(\exists x)$ as "there exist x such that $x \in D$".
- We use notation $I[\alpha]$ to represents that α is evaluated under interpretation I over a domain D.
  - $I[\alpha]$ under interpretation I over a domain D cab be evaluated to be true or false.

Interpretation – Cont…

- Let α and β are formulae and I is an interpretation over any domain D. The following holds true.
  - $I[\alpha \wedge \beta] \quad = \quad I[\alpha] \wedge I[\beta]$
  - $I[\alpha \vee \beta] \quad = \quad I[\alpha] \vee I[\beta]$
  - $I[\alpha \rightarrow \beta] \quad = \quad I[\alpha] \rightarrow I[\beta]$
  - $I[\sim\alpha] \quad = \quad \sim I[\alpha]$
- For any interpretation I and a formula using $(\forall x)$ & $(\exists x)$, the following results holds true.
  - $I[(\forall x)P(x)] = T$ iff $I[P(x)] = T, \forall x \in D$
    $= F$, otherwise
  - $I[(\exists x)\ P(x)] = T$ iff $\exists\ c \in D$ such that $I[P(c)] = T$
    $= F$, otherwise

Example - Interpretation

Let α : $(\forall x)\ (\exists\ y)\ P(x, y)$ be a formula. Evaluate α under the following interpretation I.

I

$D = \{1, 2\}$;
$I[P(1, 1)] = F$; $I[P(1, 2)] = T$;
$I[P(2, 1)] = T$; $I[P(2, 2)] = F$

Example - Cont…

Solution: Consider the following cases:

- If $x = 1$, then $\exists\, 2 \in D$ such that $I[P(1, 2)] = T$
- If $x = 2$, then $\exists\, 1 \in D$ such that $I[P(2, 1)] = T$

- Therefore, $I[\alpha] = I[(\forall x)\, (\exists\, y)\, P(x, y)] = T$ i.e., α is true under above interpretation.

Exercise

- Consider a formula α : $(\forall x)\ (P(x) \rightarrow Q(\ f(x),c))$ and the following interpretation

I

$D = \{1, 2\}$; $c = 1$; $f(1) = 2$, $f(2) = 1$
$I[P(1)] = F$, $I[P(2)] = T$
$I[Q(1, 1)] = T$, $I[Q(1, 2)] = T$,
$I[Q(2, 1)] = F$, $I[Q(2, 2)] = T$

- Find the truth value of
α: $(\forall x)\ (\ P(x) \rightarrow\ Q(\ f(x), c)\ )$ under I - (Ans: true)

Definitions

- A formula α is said to be **consistent** (satisfiable)
  - if and only if there exists an interpretation I such that $I[\alpha] = T$.
  - Alternatively, we say that I is a *model* of α or I satisfies α.
- A formula α is said to be inconsistent (unsatisfiable) if and only if
  - $\exists$ no interpretation that satisfies α or there exists no model for α.
- A formula α is valid if and only if for every interpretation I, $I[\alpha] = T$.
- A formula α is a logical consequence of a set of formulae $\{\alpha 1, \alpha 2, ..., \alpha n\}$ if and only if
  - for every interpretation I, if $I[\alpha_1 \wedge \ldots \wedge \alpha_n] = T$, then $I[\alpha] = T$.

Inference Rules in Predicate Logic

Modus Ponen Rule:

Lemma 1: If α : $(\forall x)$ ($P(x) \rightarrow Q(x)$) and β : $P(c)$ are two formulae, then $Q(c)$ is a logical consequence of α and β, where c is a constant.

Modus Tollen Rule:

Lemma 2: If α : $(\forall x)$ ($P(x) \rightarrow Q(x)$) and β : $\sim Q(c)$ are two formulae, then $\sim P(c)$ is a logical consequence of α and β, where c is a constant.

Example

- Show that δ is a logical consequence of α and β

$$\alpha \quad : \quad (\forall x)\ (P(x) \rightarrow \sim Q(x))$$
$$\beta \quad : \quad (\exists x)\ (Q(x)\ \Lambda\ R(x))$$
$$\delta \quad : \quad (\exists x)\ (R(x)\ \Lambda\ \sim P(x))$$

Solution: Let I be any interpretation over any domain D.

- Assume that I models $\alpha\ \Lambda\ \beta$ i.e., $I[\alpha \Lambda \beta] = T$ over D.
  - i.e., $I[(\forall x)\ (P(x) \rightarrow \sim Q(x))] = T$ (1)
  - and $I[(\exists x)\ (Q(x)\ \Lambda\ R(x))] = T$ (2)

Cont…

- From (2), there exist some constant $c \in D$ such that
  - $I[(Q(c) \wedge R(c))] = T$ (3)
  - i.e., $I[Q(c)] = T$ (4)
  - and $I[R(c)] = T$ (5)
- From (4),
  - $I[\sim Q(c)] = F$ (6)
- From (1),
  - $I[P(c) \rightarrow \sim Q(c)] = T$,
    where c is the same constant
  - $I[P(c)] \rightarrow I[\sim Q(c)] = T$ (7)
- From (6) and (7), we get
  - $I[P(c)] = F$
  - $I[\sim P(c)] = T$ (8)

Cont…

- From (5) and (8), we get
  - $I[R(c)] \wedge I[\sim P(c)] = T$ i.e.,
  - $I[R(c) \wedge \sim P(c)] = T$
- According to the definition of interpretation, we get
  - $I[(\exists x)(R(x) \wedge \sim P(x))] = T$ i.e.,
  - $I[\delta] = T$
- Hence,
  - δ is a logical consequence of α and β.
- This is a direct proof, often difficult.

Semantic Tableaux (Pred Logic)

- There are four more rules handling variables in a predicate formula in addition to one given for Propositional logic.
- Let us denote a formula containing a variable x by $\alpha[x]$.

Rule 10:

$$\begin{array}{|l} (\forall x)\ \alpha\,[x] \\ \alpha\,[t] \end{array}$$

for any ground term t, where t is a term free from variables.

Rules – Cont…

Rule 11: | $\sim \{(\forall x)\ \alpha\ [x]\ \}$
| $\sim \alpha\ [c]$

for any new constant c not occurring in α

Rule 12: | $(\exists\ x)\ \alpha\ [x]$
| $\alpha\ [c]$

for any new constant c

Rule 13: | $\sim \{(\exists\ x)\ \alpha\ [x]\ \}$
| $\sim \alpha\ [t]$

for any ground term t

Few Definitions

- A path in a tableaux is *contradictory* or *closed* if some atomic formulae α and ~ α appear on the same path.
- If all the paths of a tableau are closed, then it is called a *contradictory tableaux.*
- A *tableau proof* of a formula α is a contradictory tableau with root as ~ α .
- Let α be any formula. If tableaux with α as a root is a contradictory tableaux, then α is said to be *inconsistent* otherwise α is said to be *consistent*.
- A formula α is said to be *tableau provable* (denoted by |- α) if a tableau constructed with ~ α as root is a contradictory tableau.

Example

- Show that the formula

  $(\forall x)\ (P(x)\ \Lambda\ \sim (\ Q(x)\ \rightarrow P(x))\ )$

  is inconsistent.

Solution:

- We have to show that
  - tableau for $[(\forall x)\ (P(x)\ \Lambda \sim (\ Q(x)\ \rightarrow P(x)))]$ as a root is a contradictory tableau. Then by definition we can infer that the formula is inconsistent.

Example – Cont…

{Tableau root}	$(\forall x)\ (P(x)\ \Lambda \sim (\ Q(x)\ \rightarrow P(x))$	(1)
{Apply R10 on 1}	$P(t)\ \Lambda \sim (\ Q(t)\ \rightarrow P(t))$ where t is any ground term	(2)
{Apply R1 on 2}	$P(t)$ $\sim (\ Q(t)\ \rightarrow P(t))$	(3)
{Apply R7 on 3}	$Q(t)$ $\sim P(t)$	

Closed $\{P(t), \sim P(t)\}$

Soundness and completeness

Theorem: (Soundness and completeness) :

A formula α is valid ($|= \alpha$) iff α is tableau provable ($|- \alpha$).

Example: Show a validity of the following formula using tableaux method.

$$(\forall x)\, P(x) \rightarrow (\exists x)\, P(x)$$

Solution: If we show that

$\sim [(\forall x)\, P(x) \rightarrow (\exists x)\, P(x)]$ has a contradictory tableau then α is tableau provable and hence by above theorem $(\forall x)\, P(x) \rightarrow (\exists x)\, P(x)$ is valid.

More Definitions

- A set of formulae $\{ \alpha_1, \alpha_2, \ldots, \alpha_n \}$ is said to be *inconsistent* if a tableau with root as ($\alpha_1 \ \Lambda \ \alpha_2 \ \Lambda \ \ldots \ \Lambda \ \alpha_n$) is a contradictory tableau.
- *Proof of a formula* α from the set $\Sigma = \{\alpha_1, \ \alpha_2, \ldots, \alpha_n$) is a contradictory tableau with root as ($\alpha_1 \ \Lambda \ \alpha_2 \ \Lambda \ \ldots \ \Lambda \ \alpha_n \ \Lambda \sim \alpha$).
  - Alternatively, we say that α is *tableau provable* from Σ and denoted by $\Sigma \mid\!- \alpha$.
- A formula α is a *logical consequence* of Σ iff α is *tableau provable* from Σ and is denoted by $\Sigma \mid\!- \alpha$.

Exercises

I. Translate the following English sentences into Predicate Logic

- Everyone is loyal to someone.
- All Romans were either loyal to Caesar or hated him.
- For every number, there is one and only one immediate successor.
- There is no number for which 0 is immediate successor.

II. Evaluate truth values of the following formulae under the interpretation I (define your own interpretations).

- $(\exists x)\ (P(f(x)) \wedge Q(x, f(c)))$
- $(\exists x)\ (P(x) \wedge Q(x, c))$
- $(\exists x)\ (P(x) \rightarrow Q(x, c))$
- $(\forall x)(\exists y)\ (P(x) \wedge Q(x, y))$
- $(\forall x)(\exists y)\ (P(x) \rightarrow Q(f(c), y))$

III. Transform the following formulae into PNF and then into Skolem Standard Form.

- $(\forall x)(\exists y)(Q(x, y) \rightarrow P(x))$
- $(\forall x)((\exists y) P(x, y) \rightarrow \sim((\exists z) Q(z) \wedge R(x)))$
- $(\forall x)(\exists y) P(x, y) \rightarrow ((\exists y) P(x, y)$
- $(\forall x)((\exists y) P(x, y) \wedge ((\exists z) Q(z) \wedge R(x)))$
- $(\forall x)(P(x) \rightarrow Q(x)) \rightarrow ((\exists x) P(x) \rightarrow (\exists x) Q(x))$